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Solitary Waves of the Equal Width Wave Equation

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A numerical solution of the equal width wave equation, based on Galerkin's method using cubic B-spline finite elements is used to simulate the migration and interaction of solitary waves. The interaction of two solitary waves is seen to cause the creation of a source for solitary waves. Usually these are of small magnitude, but when the amplitudes of the two interacting waves are equal and opposite the source produces trains of solitary waves whose amplitudes are of the same order as those of the initiating waves. The three invariants of the motion are evaluated to determine the conservation properties of the system. Finally, the temporal evolution of a Maxwellian initial pulse is studied. For small δ ($U_t + UU_x - \delta U_{xxt} = 0$) only positive waves are formed and the behaviour mimics that of the KdV and RLW equations. For larger values of δ both positive and negative solitary waves are generated.

1. INTRODUCTION

The regularised long wave (RLW) equation is an alternative description of non-linear dispersive waves to the more usual Korteweg–de Vries (KdV) equation [1]. It has solitary wave solutions of a rather general type [1, 2]. A less well-known alternative, proposed by Morrison *et al.* [3], is the equal width equation (EWE) which also has solitary wave solutions, but of a less general type.

Solitary waves are wave packets or pulses which propagate in non-linear dispersive media. Due to dynamical balance between the non-linear and dispersive effects these waves retain a stable wave form. A soliton is a very special type of solitary wave which also keeps its waveform after collision with other solitons. In particle physics and quantum mechanics it is standard practice to use the term soliton to designate both solutions to wave equations integrable via the inverse scattering transform, such as KdV, and also to designate localised solutions of non-integrable equations, such as RLW and EWE. We have adopted the more restricted definition for which only the first type qualifies for the name soliton; thus the solitary waves of the KdV equation are solitons, but those of the RLW and EWE equations are not [3–10].

In a recent paper in the *Journal* [10] we described a Galerkin method, with cubic B-spline finite elements, which was used to obtain accurate and efficient numerical solutions to the RLW equation. Here we apply the same method to the solution of the EWE equation and study the migration of a solitary wave, the interaction of two solitary waves, and the evolution of a Maxwellian initial condition.

2. THE GOVERNING EQUATION AND FINITE ELEMENT SOLUTION

The EWE equation, derived for long waves propagating in the positive x-direction, has the form [3]

$$U_t + UU_x - \delta U_{xxt} = 0, \tag{1}$$

where δ is a positive parameter and the subscripts x and t denote differentiation, with the physical boundary condition $U \to 0$ as $x \to \pm\infty$. In this paper we shall use periodic boundary conditions for a region $a \leqslant x \leqslant b$. The form of the initial pulse will be chosen so that at large distances from the pulse $|U|$ is extremely small and essentially attains the free space boundary condition $U = 0$. In the fluid problem U is related to the vertical displacement of the water surface, while in the plasma application U is the negative of the electrostatic potential.

Although the EWE equation transforms into the RLW equation under $U_{\mathrm{EWE}} \to U_{\mathrm{RLW}} + 1$, the corresponding solutions of the EWE equation cannot be obtained from those of the RLW equation using this transformation as the boundary conditions in the two cases are different.

The EWE equation has been solved numerically using the B-spline finite element formulation described in detail in Ref. [10].

3. THE SIMULATIONS

The EWE equation has, like the RLW equation, an analytic solution of the form [3]

$$U(x, t) = 3A \operatorname{sech}^2\{k[x - x_0 - At]\} \tag{2}$$

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where

$$k = \tfrac{1}{2}\sqrt{1/\delta}, \tag{3}$$

and A is a constant. This solution corresponds to a solitary wave of magnitude $3A$ and width k, initially centred on x_0, propagating to the right without change of shape at a steady velocity A. Here k depends only on δ and not on A as does the corresponding constant for the RLW equation; thus for a given equation (fixed δ) all solitary waves have the same width, hence the name EWE. Waves exist with all possible velocities A, $-\infty \leqslant A \leqslant \infty$, unlike the RLW equation for which there is the forbidden region $0 \leqslant A \leqslant 1$. Although the solution (2) is obtained when the free space boundary condition $|U| \to 0$ as $|x| \to \infty$ is applied, it is also expected to be a very good approximation for large periodic systems.

Olver [11] has shown that the EWE equation possesses only three polynomial invariants, corresponding to conservation of mass, momentum, and energy, which for the periodic boundary conditions have the form

$$\begin{aligned} C_1 &= \int_a^b U\,dx, \\ C_2 &= \int_a^b (U^2 + \delta U_x^2)\,dx, \\ C_3 &= \int_a^b U^3\,dx. \end{aligned} \tag{4}$$

First, we consider the motion of a single solitary wave and take as initial condition

$$U(x, 0) = 3A \operatorname{sech}^2 k(x - x_0) \tag{5}$$

with $A = 1$, $k = \frac{1}{2}$, and $x_0 = 15$. The range $0 \leqslant x \leqslant 80$ is divided into 400 elements of equal length 0.2 and a time step $\Delta t = 0.1$ used. We observe the solitary wave move to the right unchanged in form and with a velocity $A = 1$.

To examine more carefully the behaviour of the numerical scheme we use the L_2 norm to compare the numerical with the exact solution (2) and the quantities C_1, C_2, and C_3 to measure conservation; see Table I. Changes in C_1, C_2, and C_3 are satisfactorily small, each changing less than $5 \times 10^{-4}\,\%$ during the experiment. The L_2 error is also small compared with values quoted by other authors for KdV simulations [12]. The change in the magnitude of the solitary wave over the period to $t = 40$ is only 0.09 %, during which time the velocity is constant at 1 to within 0.2 %. The speed and magnitude of the wave are mutually consistent with Eq. (2). Morrison *et al.* report that the primary error in their simulations is a secular drift in the wave speed which results in a change of about 1 % over a period $t = 40$ [3].

TABLE I

Single Solitary Wave Simulation

Time	C_1	C_2	C_3	$L_2 \times 10^3$	Height	Posn
0.1	12.0000	28.7999	57.5999	0.2	2.9925	15.2
2.1	12.0000	28.7998	57.5995	3.5	2.9925	17.2
36.1	11.9999	28.7997	57.5993	49.0	2.9949	51.2
40.1	11.9999	28.7998	57.5995	54.2	2.9952	55.2

Santarelli [4] has simulated the interaction of a positive and negative solitary wave, for the RLW equation, and observed the collision to produced additional pairs of daughter solitary waves emenating from the point of initial contact, an observation confirmed by Courtenay Lewis and Tjon [5] and Gardner and Gardner [10]. We now repeat those experiments for the EWE equation, with $\delta = 1$, using as an initial condition solitary waves of similar magnitudes to those used by Santarelli,

$$U(x, 0) = U_1 + U_2, \tag{6}$$

where

$$U_j = 3A_j \operatorname{sech}^2[\tfrac{1}{2}(x - \bar{x}_j - A_j t)]. \tag{7}$$

A region $0 \leqslant x \leqslant 80$ was used with $A_1 = 1.7$, $\bar{x}_1 = 23$, $A_2 = -3.4$, $\bar{x}_2 = 38$, $h = 0.1$, and $\Delta t = 0.05$.

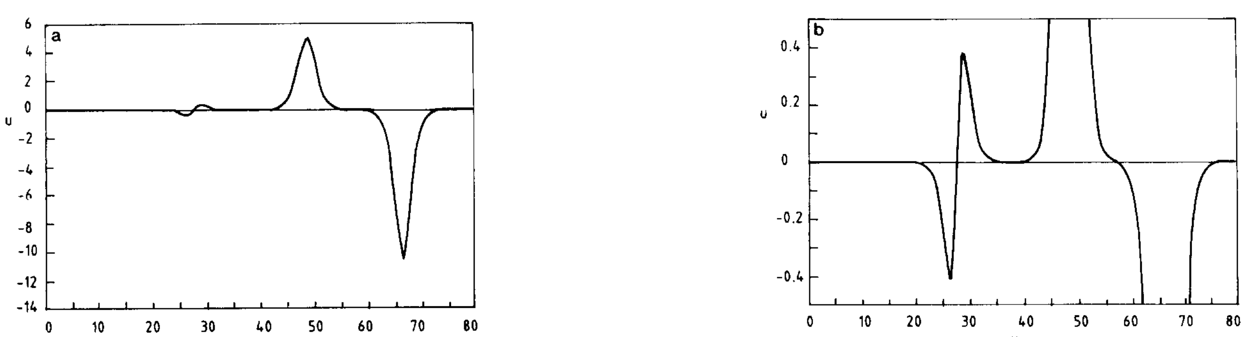

FIG. 1. (a) The "Santarelli experiment" at time $t = 16$. (b) The "Santarelli experiment" at time $t = 16$ with expanded vertical scale to show the structure of the daughter waves.

TABLE II

Santarelli Experiment

Time	C_1	C_2	C_3
0	20.400	416.16	1980.9
2	20.399	416.14	1980.8
4	20.397	416.12	1980.5
6	20.399	416.12	1980.6
8	20.398	416.11	1980.5
10	20.398	416.10	1980.4
12	20.398	416.07	1980.3
14	20.397	416.05	1980.1
16	20.397	416.05	1980.1

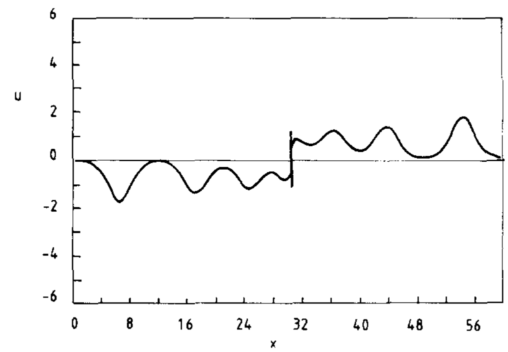

FIG. 2. The interaction of two solitary waves of equal and opposite amplitude. State at time $t = 50$.

In Fig. 1a we show the state at time $t = 16.0$. The smaller of the original pair of waves now lies at $x = 49$, the larger (negative) wave is at $x = 66$, having exited the region through the left-hand side and reentered at the right: the boundary conditions are periodic. The amplitudes of these waves are virtually unchanged by the interaction. The intervals between $0 \leqslant x \leqslant 20$ and $75 \leqslant x \leqslant 80$ are the undisturbed parts of the region, away from the pulse, where the solution remains 0. The waves lying between $x = 23$ and $x = 33$ of magnitude 0.4 have resulted from the interaction. We have expanded the vertical scale in Fig. 1b to show details of the structure of these waves. The area $x = 28$–29, where the curves cross the x-axis, is the site of the original collision. Daughter waves appear to have been created at this point. Their magnitude is small at about 0.4–0.5, so that their velocity is very small, about 0.1. At time $t = 16$ the larger (negative) wave lies at $x = 66.6$, and the smaller (positive) wave at $x = 49$. If they had progressed across the region without colliding they should have reached $x = 63.6$ and $x = 50.2$, respectively. These waves have therefore suffered phase shifts of $\Delta x = +3$ and $\Delta x = -1.2$, respectively.

The values of C_1, C_2, and C_3 throughout the simulation are shown in Table II. All change by less than 4×10^{-2}%.

We have repeated this experiment with interacting waves of equal and opposite amplitude of 4.5, within a region $0 \leqslant x \leqslant 60$, with $A_1 = 1.5$, $\bar{x}_1 = 23$, $A_2 = -1.5$, $\bar{x}_2 = 38$, $h = 0.2$, and $\Delta t = 0.1$. The interesting feature of this numerical experiment is that the invariants C_1 and C_3 corresponding to conservation of mass and energy should remain zero throughout. The state at time $t = 50$ is shown in Fig. 2, when $C_1 = 0.00022$ and $C_3 = 0.034$. The interaction of the two solitary waves of equal but opposite magnitude has resulted in the creation of a source of solitary waves, with amplitudes of the same order as the originating waves. A train of positive solitary waves of slowly decreasing amplitude is progressing to the right, while a similar train of negative solitary waves is moving left, away from the collision site. By $t = 60$, the first four waves to emerge have amplitudes of about 1.78, 1.47, 1.36, 1.15.

In several simulations we have observed the emergence of what appears to be a solitary wave source. It is known that waves are either created or absorbed at a resonance where the phase velocity (and group velocity) is zero. For the EWE which has the linear dispersion relationship $\omega(1 + k^2) = 0$ [3] these conditions are clearly possible.

Abdulloev *et al.* [2] have studied the interaction of two positive solitary waves for the RLW equation and observed an almost stationary rarefaction wave of small amplitude ($\sim 10^{-3}$) with an exceeding low velocity ($< 3 \times 10^{-5}$) left behind the two diverging solitary waves of magnitudes

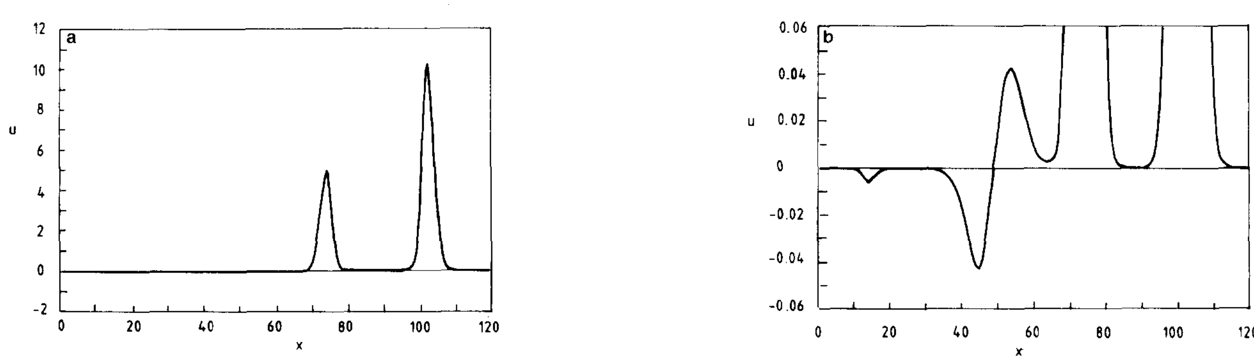

FIG. 3. (a) The "Abdulleov experiment" at time $t = 25$. (b) The "Abdulleov experiment" at time $t = 25$ with expanded vertical scale to show the trailing waves.

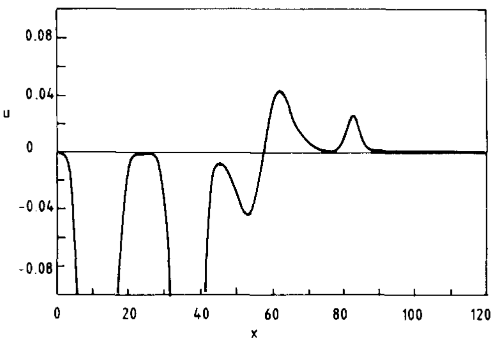

FIG. 4. The two negative waves, time $t = 20$, with trailing waves.

TABLE III

Abdulloev Experiment

Time	C_1	C_2	C_3
0	61.200	416.16	2546.9
4	61.199	416.15	2546.8
8	61.197	416.12	2546.3
12	61.197	416.12	2546.3
16	61.195	416.10	2546.3
20	61.195	416.07	2546.0
24	61.190	416.04	2545.8
25	61.190	416.04	2545.7

about 6 and 2. We study a similar situation using as the initial condition Eqs. (14)–(15) over the region $0 \leqslant x \leqslant 120$ taking $\delta = 1$ $A_1 = 3.4$, $\bar{x}_1 = 15$, $A_2 = 1.7$, $\bar{x}_2 = 35$, $h = 0.1$, and $\Delta t = 0.05$.

The configuration at time $t = 25$, which is sometime after the interaction is complete, is shown in Fig. 3a. The waves have apparently passed through one another and emerged unchanged by the encounter. Phase shifts of $\Delta x = +3$, for the larger wave, and of $\Delta x = -3$, for the smaller wave, were obtained. Under magnification, however, Fig. 3b, we observe waves of small amplitude, average $\sim 5 \times 10^{-2}$, trailing behind the solitary waves. We believe that this collision too has resulted in the creation of a solitary wave source sited at $x = 49$.

In Table III we record the values of the invariants C_1, C_2, and C_3 for times throughout the simulation. We see that each is satisfactorily conserved, as each changes by less than $5 \times 10^{-2}\%$ during the computer run.

In addition we have studied the interaction of two negative solitary waves over the region $0 \leqslant x \leqslant 120$, using the previous initial condition with $\delta = 1$, $A_1 = -3.4$, $\bar{x}_1 = 82$, $A_2 = -1.7$, $\bar{x}_2 = 67$, $h = 0.1$, and $\Delta t = 0.05$.

After the interaction is completed the two solitary waves have changed little but there is some evidence of an additional disturbance. Under magnification, Fig. 4, we see that waves of small amplitude $\sim 5 \times 10^{-2}$ are trailing behind the solitary waves. The larger wave has suffered a phase change of $\Delta x = +2$, while the smaller has a phase change of $\Delta x = -3.5$.

The values taken by the invariants C_1, C_2, and C_3 over the period of simulation are given in Table IV. All are satisfactorily conserved; C_1 changes by less than $9 \times 10^{-2}\%$, C_2 by less than 0.2%, and C_3 by less than 0.26%. These values are of the same order as those found for the Santarelli experiment.

Finally, we have examined, for various values of the parameter δ, the evolution of an initial Maxwellian pulse into solitary waves, using as initial condition

$$U(x, 0) = \exp(-(x-7)^2). \tag{8}$$

For the KdV equation this initial pulse developed into a train of solitons when δ was greater than a critical value $\delta_{cr} = 0.0625$; otherwise a rapidly oscillating wave packet resulted [13]. In the case of the RLW equation similar events were observed but the development was never clean and oscillating tails of small magnitude always trailed behind the solitary waves [10].

With $\delta = 0.01$ the final state is made up from at least seven solitary waves, Fig. 5; the peaks of the well-developed waves lie on a straight line so that their velocities are linearly dependent on their amplitudes and, in fact, obey a relationship consistent with Eq. (2). On magnification of

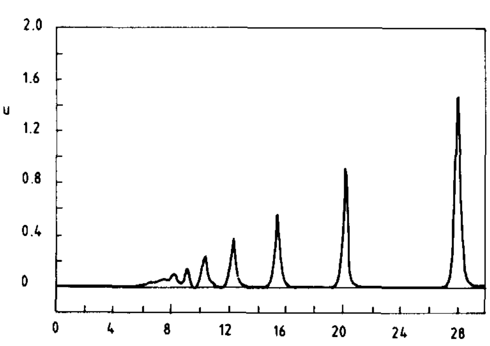

FIG. 5. Maxwellian initial condition: $\delta = 0.01$; state at time $t = 41$.

TABLE IV

Two Negative Solitary Waves

Time	C_1	C_2	C_3
0	61.20	416.2	2547
4	61.19	416.0	2545
8	61.18	416.0	2541
12	61.17	415.8	2543
16	61.16	415.7	2542
20	61.15	415.5	2540

TABLE V

$\delta = 0.01$

Time	C_1	C_2	C_3
0.1	1.773	1.266	1.023
4.1	1.772	1.259	1.064
8.1	1.770	1.256	1.067
12.1	1.769	1.254	1.064
20.0	1.767	1.250	1.048
31.0	1.763	1.244	1.049
41.0	1.761	1.240	1.050
51.0	1.758	1.236	1.035

TABLE VI

$\delta = 0.04$

Time	C_1	C_2	C_3
0.10	1.7725	1.3034	1.0233
10.1	1.7724	1.3019	1.0262
20.0	1.7723	1.3018	1.0260
30.0	1.7723	1.3016	1.0260
50.0	1.7721	1.3015	1.0257

TABLE VII

$\delta = 0.2$

Time	C_1	C_2	C_3
0.1	1.77245	1.50391	1.02333
4.1	1.77246	1.50383	1.02338
8.1	1.77245	1.50388	1.02336
10.1	1.77246	1.50388	1.02335
20.0	1.77246	1.50388	1.02335
30.0	1.77246	1.50387	1.02335
40.0	1.77246	1.50387	1.02335

TABLE VIII

$\delta = 1$

Time	C_1	C_2	C_3
0.15	1.77245	2.5048	1.0233
12.15	1.77246	2.5051	1.0231
24.15	1.77245	2.5060	1.0231
30	1.77245	2.5060	1.0231
42	1.77245	2.5060	1.0231
54	1.77245	2.5060	1.0231
60	1.77245	2.5060	1.0231

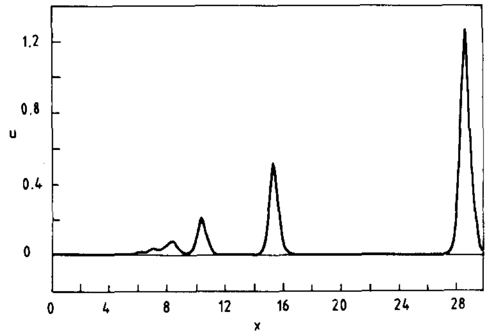

FIG. 6. Maxwellian initial condition: $\delta = 0.04$, state at time $t = 49$.

the vertical scale no oscillating tail is evident. The invariants for this problem are listed in Table V. Observed changes are each less than 5% for a run up to $t = 51$.

For $\delta = 0.04$ the Maxwellian develops into a train of at least four solitary waves with magnitude and velocity consistent with Eq. (2), but again no oscillating tail is observed; see Fig. 6. The values of the quantities C_1, C_2, and C_3 are given in Table VI; each is satisfactorily constant as the maximum change is less than 0.01%.

When $\delta = 0.2$, a different behaviour is observed. The state at time $t = 40$ is shown in Fig. 7 with a magnified vertical scale. We see an isolated disturbance of small magnitude ~ 0.03 trailing behind the solitary wave which has a measured velocity of 0.34 ± 0.004 and an amplitude of 1.024 which implies a theoretical velocity of 0.341, in good agreement with the measured value. The invariants C_1, C_2, and C_3 are given in Table VII; all vary by less than 0.005% over the period to $t = 40$.

For values of δ greater than a critical value, which for the KdV equation is $\delta = 0.0625$ and for the RLW equation is somewhat larger, solitary waves are not expected [13]. We have set up a simulation for $\delta = 1$ which for both KdV and RLW equations results in the development of a rapidly oscillating wave packet. In the present case we find that the Maxwellian develops into a pair of solitary waves, one of which has a negative amplitude -0.359 and velocity

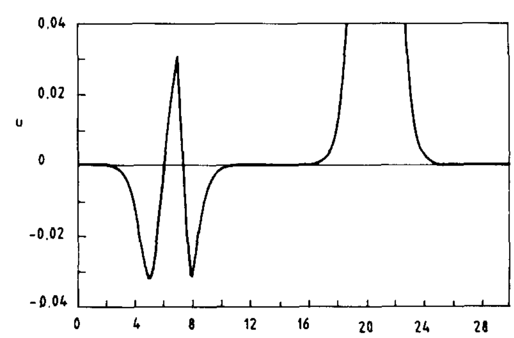

Fig. 7. Maxwellian initial condition: $\delta = 0.2$, state at time $t = 40$ with expanded vertical scale to show the trailing waves.

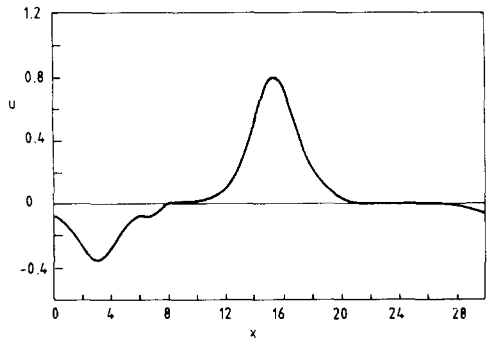

FIG. 8. Maxwellian initial condition: $\delta = 1$, state at time $t = 36$.

-0.115 ± 0.004, while the other has amplitude 0.807 and velcity 0.265 ± 0.004, each of which is consistent with Eq. (2). There is evidence that additional small waves may occur between the two main solitary waves; see Fig. 8. The invariants C_1, C_2, and C_3 remain constant to within 0.05% throughout the run; see Table VIII.

4. CONCLUSIONS

We have shown that the Galerkin method with B-spline finite elements can faithfully represent the amplitude, position, and velocity of a single solitary wave. The interaction of two solitary waves appears to cause the creation of a source of both positive and negative solitary waves. The three invariants of motion are sensibly constant in all the computer simulations described here, so that the algorithm can fairly be described as conservative. We have further used the algorithm to simulate the generation of EWE-solitary waves from a Maxwellian initial pulse. We find that the behaviour is significantly different from that of both the KdV and RLW equations. For $\delta = 0.01$ and 0.04 the Maxwellian initial condition develops into a train of positive solitary waves mimicking the behaviour of the KdV equation, and unlike the RLW equation no oscillating tail is observed. This is consistent with the observation by Morrison *et al.* [3] that EWE does not have radiation solutions, which implies that all solutions are solitary waves. When $\delta = 0.2$ we obtain a single large solitary wave, plus what appears to be a source of both positive and negative solitary waves of very small amplitude. For $\delta = 1$ both the KdV and RLW equations develop, not solitary waves but, rapidly oscillating wave packets. For the EWE equation we find that a solitary wave of amplitude ~ 0.8 plus one of amplitude ~ -0.4 is generated, together with what appears to be a source of small amplitude solitary waves.

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